Jackson 14.1 Lienard Wiechert Potentials

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This paper attempts to convey the content of Jackson's Electrodynamics Ch 14.1 in a thorough and perspicuous manner.

PACS numbers:

I. INTRODUCTION

In Jackson 14.1, Jackson addresses the question of, "What are the potentials and fields for a moving point charge?" This is obviously a fundamental question, which has surprisingly not been answered thusfar in our study of Electrodynamics. The difficulty arises in that the point source cannot strictly be considered a steady current nor obviously can it be considered stationary. These constraints draw the point-charge from the tenable regions of electrostatics into the hairy and often intractable domain of relativistic electrodynamics. Intractable, though it may often be, Jackson approaches this task with the full engine of Lorentz Covarient 4-Potential and arrives, via some dextrous math-calisthetics to the famous Leinard Weichert Potentials. These potential equations should form the backbone for anyone's study of electrodynamics, since they fully describe the the potentials and hence the fields of a moving point charge. From those equations, a physicist can tackle many practical problems faced in laboratories or even extend the theory to predict the fields of more extensive bodies. However, as impressive and general as Jackson's derivation of these formulas is, I will first entertain a more whimsical derivation, which although may not be as versatile, it gives keen insight into the behavior of the equations themselves.

II. AN INTUITIVE DERIVATION

If you can imagine for a second, an electron very far away moving very fast. Perhaps, this electron was birthed in a fiery ionosphere and is now being swept away by the solar winds into the dark reaches of space. Or, perhaps it's a more pedestrian gedanken-electron, wafting through the emptiness of our imaginations. Whatever the case, it's moving quickly and it's quite far away. Since it is far away and moving quickly, it would be naive of us to neglect the fact that the particle we are seeing is merely the image of the particle from an earlier time. Further more, the field that we are feeling is simply a residual field from the electron at an earlier time as well. Taking this into account, we can calculate the potential for this electron - simply by using the position of the electron as we 'see' it - a ghost of an electron's past.

$$\phi(x,t) = \int \frac{\rho(x',t_r)}{R} dV' \tag{1}$$

FIG. 1: Where a train was when the light left the rear of the engine, and where the the train is when the light reaches the front of the engine. The length of the train is: L, the length of the train and travel distance is L'.

Where we simply remember that the R in the equation is actually pointing to the image of the electron's passing,

$$\left|\vec{x}(t) - \vec{r}(t_r)\right| = c(t - t_r) \tag{2}$$
$$\vec{x}(t) - \vec{r}(t_r) = \vec{R} \tag{3}$$
$$\tag{4}$$

So, using our cunning understanding of the propogation of information as well as some witty analogies about ghosts, we can come to the easy conclusion that,

$$\phi(x,t) = \frac{e}{R}. \tag{5}$$

Since we are talking about a single electron, instead of a distributoin, the integral actually simplifies to a single term - for a single electron! This however, is an unfortunate conclusion, because it's wrong. There was an insidious subtlety that we tripped over in our rush to pat ourselves on the back.

$$\int \rho(x',t_r) dV' \neq e \tag{6}$$

Any resonable person, with any knowledge of an electron would certainly be baffled at this point. How can the density of an electron change? The electron doesn't have any known volume, so that cannot change. Certainly, the hypothetical idealized electron has no volume either! Charge conservation is a powerful conservation, certainly we haven't misplaced some of our charge. So how could the density of an electron change? The answer is that we are not actually summing up the fields due to electrons, but rather the fields due to the images of electrons - and although the electron itself remains pure, it's image is more susceptible to the trickeries and vagaries of a carnival funhouse - the image can be and definitely is distorted by the relative motion between the electron and the observer. This is purely a geometric effect. This can be more clearly explained using an object which has a bit more volume than a single electron, like say, a train.

Consider for a moment, train coming toward you, much like in the figure, Fig. 1. Light leaving the rear of the train and travels for a certain time, Δt, before reaching the head of the train. In that time frame, the train head of the train has traveled a distance,

$$v(\Delta \mathrm{t}) = L' - L. \tag{7}$$

In that same time frame, the light from the rear of the train has traveled a distance,

$$c(\Delta \mathrm{t}) = L'. \tag{8}$$

Once the light from the rear of the train has reached the front of the train, it meets up with a photon from the front of the train, and they travel together - happily into the sunset to form an image together. The catch is that the image is distorted by this difference in travel times, so that the apparent length of the train can be found by,

$$\frac{L' - L}{v} = \frac{L'}{c} \tag{9}$$

$$L' = L\left(\frac{1}{1 - \left(\frac{v}{c}\right)}\right) \tag{10}$$

Which, easily generalizes to a volume.

$$V' = V\left(\frac{1}{1 - \vec{\beta} \bullet \hat{n}}\right). \tag{11}$$

A key point to this generalization is that it does not reference specifically what the volume is, it could be anything as large as a train... as small as a breadbox... or even, as small as an electron. So, if the apparent volume of our electron distorts by a factor of $1/(1 - \vec{\beta} \cdot \hat{n})$. Then the density must become visually distorted as well, giving a scalar potential of,

$$\phi(x, t) = \int \frac{\rho(x', t_r)}{R} dV' = \frac{1}{R}\left(\frac{e}{1 - \vec{\beta} \bullet \vec{n}}\right). \tag{12}$$

Likewise, we can find the vector potential, $\vec{A}(x, t)$

$$\vec{A}(x, t) = \int \frac{\rho(x', t_r)\vec{v}(t_r)}{R} dV' \tag{13}$$

$$\vec{A}(x, t) = \frac{\vec{v}(t_r)}{R} \int \rho(x', t_r) dV' \tag{14}$$

$$\vec{A}(x, t) = \frac{\vec{v}(t_r)}{R}\left(\frac{e}{1 - \vec{\beta} \bullet \hat{n}}\right) \tag{15}$$

The equations make very obvious sense. Simply use the fundamental formulas for the potentials, remembering to take into account both the travel time for the 'information' to leave the source charge and also take into account the geometric distortion of the volume due to its motion. Now that we have seen how simple this can be, it's time to look at the alternative.

III. JACKSON'S DERIVATION

Jackson's derivation of the same material is daunting to say the least. He chose to begin with a very abstract and generalized set of equations. The depth of his abstraction, obfuscated the physics behind the original equations as well as the subsequent mathematical steps. He chose to begin chapter 14.1 with,

$$A^{\alpha}(x) = \frac{4\pi}{c} \int G_r(x, x') J^{\alpha}(x') d^4x' \tag{16}$$

$$J^{\alpha}(x') = \mathrm{ec} \int U^{\alpha}(\tau) \delta^4(x_{\beta}' - r(\tau)_{\beta}) d\tau \tag{17}$$

$$G_r(x, x') = \frac{H(x_0 - x_0{}')}{2\pi} \delta(\Delta \mathrm{x}_{\alpha} \Delta \mathrm{x}^{\alpha}) \tag{18}$$

Here, he is choosing to work with the 4-vector, 4-current, and the mysterious 'retarded' Green's Function, G_r. Comparing the simpler, more familiar, and less intimidating versions of the same equations.

$$\vec{A}(x, t) = \int \frac{\rho(x', t_r)\vec{v}(t_r)}{R} dV' \tag{19}$$

$$\phi(x, t) = \int \frac{\rho(x', t_r)}{R} dV' \tag{20}$$

Is apt to give one the chills, since there is so little resemblence between the two. One set works with the traditional 3-vectors with which we are very comfortable with, the other set works in the etheral realm of 4-vectors. The most intimidating math in one set, is an integral over a charge distribution, the other set requires one to tackle the abstraction of a Green's Fuction. A 'retarded' one at that. Certainly, before we can go anywhere, we should spend a moment reflecting on the meaning of the Green's Fuction.

IV. GREEN'S FUNCTION

The Green's Function is a large subject in and of itself and could alone be the subject of several larger papers. I spent most my talk going over this subject, because I thought it would be the most helpful thing I could do for the my classmates. However, here, I will settle for a terser heuristic explaination along with some simple mathematical definitions.

When working with linear differential equations, as we often do in physics, it's possible to superimpose two known solutions to generate a 3rd as of yet uknown solution to the differential equation. For example, were I to know that A was the solution to:

$$y'' + by' = 5 \tag{21}$$

and that, B was the solution to:

$$y'' + by' = 3 \tag{22}$$

I would also then know that that A + B would be the solution to:

$$y'' + by' = 8 \tag{23}$$

Because, solutions add linearly. This example was quite trivial, however, in principal it represents the heart of what a Green's Function is. The Green's Function is an attempt to build solutions to inhomogeneous linear differential equations from a fundamental solution ... the solution to the differential equation subject to a delta function impulse. Mathematically speaking, this is done by summing up the delta-function solutions until you have built the function of interest,

$$A = \int G(x, x')F(x')dx' \tag{24}$$

The only thing that one must do, is find the Green's Function for a given differential equation and boundary conditions. In general, this is not an easy task. Normally it is done by a combination of expanding the function in terms of eigenfunctions, integral transformations, and the division of region techniques. Later in this paper, we will actually go through the process of finding a Green's Function for a more simple physical system. The arduousness of this task, leaves one to wonder how difficult it can be in general to find the Green's Function.

In a somewhat dreamy manner, one might be inclined to describe the Electrodynamic Green's Function as the etheral response of the cosmos to the electrons whimsical strummings.

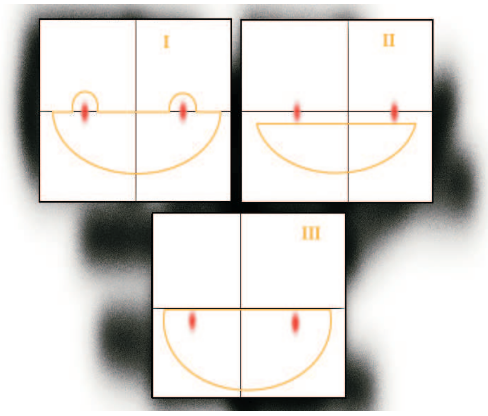

FIG. 2: Three possible contours that can be used to evaluate this integral. I chose to work with number 3.

V. THE ORIGIN OF EQN 14.1

Now that we are slightly warmed to the idea of what a Green's Function is, we turn our attention to the origins of these equations,

$$A^{\alpha}(x) = \frac{4\pi}{c}\int G_r(x, x')J^{\alpha}(x')d^4x', \tag{25}$$

$$J^{\alpha}(x') = \mathrm{ec}\int U^{\alpha}(\tau)\delta^4(x_{\beta}' - r(\tau)_{\beta})d\tau, \tag{26}$$

$$G_r(x, x') = \frac{H(x_0 - x_0')}{2\pi}\delta(\Delta \mathrm{x}_{\alpha}\Delta \mathrm{x}^{\alpha}). \tag{27}$$

The humble origins of these super-equations begin long ago, deep in the reaches of Jackson, Chapter 12, under the guise of Maxwell's Equations. Here, Jackson was attempting attempting to solve the covariant Maxwell's Equations in open space by developing a covarient Green's Fuction.

He begins here, with the covarient formulation of Maxwell's Equations,

$$\partial_{\alpha}F^{\alpha\beta} = \frac{4\pi}{c}J^{\beta}. \tag{28}$$

Where, $F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}$. From this identity, we can proceed to find the differential equations that we will be attempting to solve,

$$\partial_{\alpha}\left[\partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}\right] = \frac{4\pi}{c}J^{\beta} \tag{29}$$

$$\partial_{\alpha}\partial^{\alpha}A^{\beta} - \partial_{\alpha}\partial^{\beta}A^{\alpha} = \frac{4\pi}{c}J^{\beta} \tag{30}$$

$$\Box \mathrm{A} - \partial_{\alpha}\partial^{\beta}A^{\alpha} = \frac{4\pi}{c}J^{\beta} \tag{31}$$

In open space, we expect our equations to satisfy the Lorenz condition,

$$\frac{1}{c}\frac{\mathrm{d}\phi}{\mathrm{dt}} + \vec{\nabla} \bullet \vec{A} = 0 \tag{32}$$
$$\partial_\beta A^\beta = 0. \tag{33}$$

This leaves us at the 4-dimensional Poisson equation,

$$\Box \mathrm{A} = \frac{4\pi}{c} J^\beta. \tag{34}$$

Which, as we know from our discussion on Green's Functions, a solution can be constructed by superimposing delta-function solutions in such a way to mimick our source term,$4\pi J^\beta/c$. To begin this, we will first have to find the system's response to the delta-function input.

$$\Box G = \delta^4(r_\alpha - r'_\alpha) \tag{35}$$

Here, the 4-dimensional delta function can be re-expressed, as $\delta^4(r_\alpha - r'_\alpha) = \delta(x_0 - x'_0)\delta^3(\vec{r} - \vec{r}\,')$, and G is actually a function of, $G(x_0, x_1, x_2, x_3)$.

When attempting to find the Green's Function for a differential equation, it is often wise to Fourier transform as often as possible. This changes the differential equation (in this case a partial differential equation)into a more manageable algebraic equation. In the best case scenario, the differential equation can be solved through algebra alone in the transformed space. In which case (and this case is one such case) you simply will need to transform back after performing the algebra and you will have your Green's Function in hand. Following Jackson's lead, we begin by transforming the variables one at a time,

$$\tilde{G}(k_0, x_1, x_2, x_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{G} e^{i\mathrm{k}_0 x_0} dx_0 \tag{36}$$

Where, $\tilde{G}(k_0, x_1, x_2, x_3)$ is the transformation of a single coordinate of $G(x_0, x_1, x_2, x_3)$ from 'coordinate space' to 'phase space.' If we continue to compound the process through each of the variables, we quickly find that they transform independently,

$$\tilde{\tilde{G}}(k_0, k_1, x_2, x_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{G} e^{-i\mathrm{k}_1 x_1} dx_1 \tag{37}$$
$$\tilde{G}^3(k_0, k_1, k_2, x_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\tilde{G}} e^{-i\mathrm{k}_2 x_2} dx_2 \tag{38}$$
$$\tilde{G}^4(k_0, k_1, k_2, k_3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{G}^3 e^{-i\mathrm{k}_3 x_3} dx_3 \tag{39}$$

I denoted additional compounded transformations by placing the number of transformations as the superscript. Finally, after reaching the end with $\tilde{G}^4$, I am re-naming it to be the total fourier transformation of all variables,

$$\tilde{G}(k_0, k_1, k_2, k_3) = \frac{1}{(2\pi)^2} \int G e^{i\mathrm{k}_\alpha x^\alpha} dV. \tag{40}$$

You may have noticed that the sign in the exponent changed between the 0^{th} components and the rest. That's because, when working with 4-vectors, the projection along the spatial coordinates is oppositely signed from the time coordinate. This all comes together nicely in the final equation. We have shown how G transforms, but that is not quite what we need to evaluate, $\Box G = \delta^4(r_\alpha - r'_\alpha)$. We need to know how to transform the derivatives of G. Well, to do that we look at how to express G in terms of the reverse fourier transform,

$$G = \frac{1}{(2\pi)^2} \int \tilde{G} e^{-i\mathrm{k}_\alpha x^\alpha} dV_k. \tag{41}$$

Then, operate on G, with a derivative. The derivative is in x, but the integral is over k-space. So the derivative passes through the integral without any problems.

$$\frac{dG}{dx_i} = \frac{1}{(2\pi)^2} \int \frac{d}{dx_i} \tilde{G} e^{-i\mathrm{k}_\alpha x^\alpha} dV_k. \tag{42}$$

This simply produces a constant, k_i from the exponential term. Reverse transforming both sides tells us that each differentiation will simply produce a constant, k.

$$\frac{d^2 G}{\mathrm{dx_0}^2} = (-i\mathrm{k}_0)(-i\mathrm{k}_0)\tilde{G} = -{k_0}^2 \tilde{G} \tag{43}$$
$$\frac{d^2 G}{\mathrm{dx}_i^{\,2}} = -k_i k^i \tilde{G} \tag{44}$$

This quickly leads to the fourier transformed differential equation,

$$-k_i k^i \tilde{G} = \frac{1}{(2\pi)^2} e^{-i\mathrm{k}_\alpha (0)} \tag{45}$$

This leads immediately to the solution, for $\tilde{G}$:

$$\tilde{G} = \frac{1}{(2\pi)^2} \frac{-1}{k_\alpha k^\alpha}. \tag{46}$$

Which in turn, immediately leads to the solution for G, via the inverse transformation.

$$G = \frac{1}{(2\pi)^4} \int \frac{-1}{k_\beta k^\beta} e^{-i\mathrm{k}_\alpha x^\alpha} dV_k \tag{47}$$

As one almost always done, when attempting to return from fourier-space, one boards the rocketship "Contour Integration" and buckles up for a ride. First we have to change from speaking about 'real' k to speaking about 'complex' k. So henceforth, while going through this contour integral, k will be understood to be a complex number. Step two, is to make the poles more conspicuous.

$$k_\beta k^\beta = 0 \tag{48}$$
$$k_0{}^2 = \vec{k} \bullet \vec{k} \tag{49}$$
$$G = \frac{1}{(2\pi)^4} \int \int \frac{-1}{k_0{}^2 - \vec{k} \bullet \vec{k}} e^{-i k_0 x_0} e^{i \vec{k} \bullet \vec{x}} dk_0 d^3k \tag{50}$$

With the equation written more explicitly on the last line, we see a slight problem with our equation. It has two poles laying on the real axis. This means that we cannot simply choose the real axis as one of our contours for our path. We have 3 options available to us. We can distort our contour around the poles (creating little semi-circles) and solve for the Cauchy Principal Value. This sometimes gives the correct answer, and I believe it does in this case in particular. Or, we can do as Jackson does, he chooses a contour infinitely close to the real axis, thus narrowly avoiding the poles. Or, equivalently, you can nudge the poles off of the real axis by some small imaginary number. This last option is the option that I chose to do, Fig. 2. I had read, but did not understand why, that closing the contour down preserves causality and produces the 'retarded' Green's Function. In order to do the integral this way, I had to first shift the poles down.

$$f = \frac{-1}{(k_0 - |k| + i\epsilon)(k_0 + |k| + i\epsilon)} \tag{51}$$
$$G = \frac{1}{(2\pi)^4} \int \int f e^{-i k_0 x_0} e^{i \vec{k} \bullet \vec{x}} dk_0 d^3k \tag{52}$$

This gives us two poles at,

$$k_0 = |\vec{k}| - i\epsilon \tag{53}$$
$$k_0 = -|\vec{k}| - i\epsilon \tag{54}$$

And from there, we should all remember the fundamental rule for contour integration:

$$\int_c f e^{-i k_0 x_0} dk_0 = 2\pi i \sum f_{residues} \tag{55}$$

For some strange reason, Jackson works with a different convention, $-2\pi i$. I chose to proceed with the convention that I knew. The poles are simple poles, so we can immediately find the residues,

$$\frac{-(k_0 + |k| + i\epsilon) e^{-i k_0 x_0}}{(k_0 + |k| + i\epsilon)(k_0 - |k| + i\epsilon)} \text{with} : k_0 = -|k| - i\epsilon \tag{56}$$
$$\frac{-(k_0 - |k| + i\epsilon) e^{-i k_0 x_0}}{(k_0 + |k| + i\epsilon)(k_0 - |k| + i\epsilon)} \text{with} : k_0 = |k| - i\epsilon \tag{57}$$

Where in the limit as ϵ goes to zero:

$$\frac{lim}{\epsilon \to \infty} = \sum f = \frac{-e^{-i|k|x_0}}{2|k|} + \frac{e^{i|k|x_0}}{2|k|} \tag{58}$$

From there, it's a simple simplification to find:

$$\sum f = \frac{i}{|k|} \sin(|k| x_0) \tag{59}$$

Now that we have the residue for our function in hand, we vist the contour integral. It's important to remember that we have found the integral over the entire contour. We are interested in finding the integral across a single leg of the countour (the real axis).

$$\int_c f e^{-i k_0 x_0} dk_0 = 2\pi i \Big(\frac{i}{|k|} \sin(|k| x_0) \Big) \tag{60}$$
$$\int_c f e^{-i k_0 x_0} dk_0 = \frac{2\pi}{|k|} \sin(|k| x_0) \tag{61}$$
$$\int_c f e^{-i k_0 x_0} dk_0 = \int_{c_r} f e^{-i k_0 x_0} dk_0 + \int_{-\infty}^{\infty} f e^{-i k_0 x_0} dk_0 \tag{62}$$

Here, c is the entire contour. c_r is the leg of the countour extending in a half-circle at infinity, and what is left over is the real-axis leg. Since the modulus of our function goes to zero as the modulus of k goes to infinity,

$$\frac{lim}{k_0 \to \infty} \Big(\frac{1}{k_0{}^2} \Big) = 0 \tag{63}$$

We can conclude that our contribution from our large loop is:

$$\int_{c_r} f e^{-i k_0 x_0} dk_0 = 0 \tag{64}$$

So we know the contribution from the real leg must be, (because the only other branch vanished!):

$$\int_{-\infty}^{\infty} \frac{-1}{k_0{}^2 - \vec{k} \bullet \vec{k}} e^{-i k_0 x_0} dk_0 = \frac{2\pi}{k} \sin(\mathrm{kx_0}) \tag{65}$$

Plugging this back into our original 4-dimensional integral for G,

$$G = \frac{1}{(2\pi)^4} \int \frac{2\pi}{k} \sin(\mathrm{kx}_0) e^{i\vec{k}\cdot\vec{x}} d^3k \quad (66)$$

Expanding the differential volume element, and writing the integrands explicitly,

$$G = \frac{H(x_0)}{(2\pi)^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \frac{\sin(\mathrm{kx}_0)}{k} e^{i\vec{k}\cdot\vec{x}} k^2 \sin(\theta) dk d\theta d\phi \quad (67)$$

H is the heavy-side step function, using the symbol H instead of θ to avoid confusion with the coordinate. If we take the polar axis of our cordinate system to be along x, $\vec{k}\bullet\vec{x} = \mathrm{kx}\cos(\theta)$, and we magically guess the following identity:

$$\frac{d}{d\theta}\left[\frac{e^{i\vec{k}\cdot\vec{x}}}{i\mathrm{kx}}\right] = \left[\frac{d\vec{k}\bullet\vec{x}}{d\theta}\right] \frac{d\left[\frac{e^{i\vec{k}\cdot\vec{x}}}{i\mathrm{kx}}\right]}{d\left(\vec{k}\bullet\vec{x}\right)} \quad (68)$$

At this point, I am adopting the Jackson sign convention for the residue and proceeding systematically through the integration:

$$\frac{d}{d\theta}\left[\frac{e^{i\vec{k}\cdot\vec{x}}}{i\mathrm{kx}}\right] = \frac{i e^{i\vec{k}\cdot\vec{x}}}{i\mathrm{kx}} \mathrm{kx}\sin(\theta) = e^{i\vec{k}\cdot\vec{x}}\sin(\theta) \quad (69)$$

$$G = \frac{H(x_0)}{(2\pi)^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} k\sin(\mathrm{kx}_0) \frac{d}{d\theta}\left[\frac{e^{i\vec{k}\cdot\vec{x}}}{i\mathrm{kx}}\right] dk d\theta d\phi \quad (70)$$

$$G = \frac{H(x_0)}{(2\pi)^3} \int_0^{2\pi} \int_0^{\infty} k\sin(\mathrm{kx}_0) \left[\frac{e^{i\vec{k}\cdot\vec{x}}}{i\mathrm{kx}}\right]_0^{\pi} dk d\phi \quad (71)$$

$$G = \frac{H(x_0)}{(2\pi)^3} \int_0^{2\pi} \int_0^{\infty} k\sin(\mathrm{kx}_0) \left[\frac{e^{i\mathrm{kx}}}{i\mathrm{kx}} - \frac{e^{-i\mathrm{kx}}}{i\mathrm{kx}}\right] dk d\phi \quad (72)$$

$$G = \frac{H(x_0)}{(2\pi)^3} \int_0^{2\pi} \int_0^{\infty} \sin(\mathrm{kx}_0) \left[\frac{2\sin(\mathrm{kx})}{x}\right] dk d\phi \quad (73)$$

$$G = \frac{H(x_0)}{(2\pi)^2} \int_0^{\infty} \sin(\mathrm{kx}_0) \left[\frac{2\sin(\mathrm{kx})}{x}\right] dk \quad (74)$$

$$G = \frac{H(x_0)}{2\pi^2 x} \int_0^{\infty} \sin(\mathrm{kx}_0)\sin(\mathrm{kx}) dk \quad (75)$$

It was slightly unfair that we had to mystically guess the identity to proceed in the above. But, having guessed it, we have were able to proceed in a fairly straightforward manner to the result of Jackson 12.130. And we only have one integral left over, the easiest of the group. Using the identity, $\sin(z) = \frac{e^{iz}}{2i} - \frac{e^{-iz}}{2i}$, we begin dismantling the integral.

$$G = \frac{H(x_0)}{2\pi^2 x} \int_0^{\infty} \left(\frac{e^{i\mathrm{kx}_0}}{2i} - \frac{e^{-i\mathrm{kx}_0}}{2i}\right)\left(\frac{e^{i\mathrm{kx}}}{2i} - \frac{e^{-i\mathrm{kx}}}{2i}\right) dk \quad (76)$$

$$= \frac{-H(x_0)}{8\pi^2 x} \int_0^{\infty} \left(e^{i\mathrm{kx}_0} - e^{-i\mathrm{kx}_0}\right)\left(e^{i\mathrm{kx}} - e^{-i\mathrm{kx}}\right) dk \quad (77)$$

$$= \frac{-H(x_0)}{8\pi^2 x} \int_0^{\infty} e^{i\mathrm{k}(x_0+x)} - e^{i\mathrm{k}(x_0-x)} - e^{i\mathrm{k}(x-x_0)} + e^{-i\mathrm{k}(x+x_0)} dk \quad (78)$$

Now, we are on the cusp of a key step. Notice that we are now changing the boundaries of integration. This is to get the integral-form of the delta function. We can only do this because x is constrained to be positive by the step function, and k is strictly positive as well. So we have both halves of the total integral from $-\infty$ to ∞ just laying there, waiting to be assembled.

$$G = \frac{-H(x_0)}{8\pi^2 x} \int_{-\infty}^{\infty} e^{i\mathrm{k}(x_0+x)} - e^{i\mathrm{k}(x_0-x)} dk \quad (79)$$

$$G = \frac{-H(x_0)}{8\pi^2 x} 2\pi(\delta(x_0 + x) - \delta(x_0 - x)) \quad (80)$$

$$x_0 \& x > 0 \quad (81)$$

$$G(x) = \frac{H(x_0)}{4\pi \mathrm{x}} \delta(x_0 - x) \quad (82)$$

$$G(x, x') = \frac{H(x_0 - x_0{}')}{4\pi \mathrm{x}} \delta(x_0 - x_0{}' - x) \quad (83)$$

And finally, we have it! 55 Math operations later, we have found the Green's Function for Maxwell's Equations in open space! Notice that it is symmetric in the primed and unprimed coordinates, that's required when you have no boundaries. At this juncture I would probably be satisfied, Jackson however, being a calculating machine, presses onward to find the Lorentz Covariant form of the Green's Function. This is actually a really important step, if we want to preserve the generality of our 4-vectors. So, let's put our trooper-shoes back on.

First of all, let me introduce an identity for delta functions,

$$\delta(g(x)) = \sum \frac{\delta(x_{\mathrm{io}})}{|g'(x_{\mathrm{io}})|} \quad (84)$$

This identity will crop up several more times. The conditions of this equality are that there are a finite number for zeros to g(x) and that g(x) contains no double roots. Because, if it does contain a double root, then the denominator will be zero.

As I was saying before, we wish to rewrite the Green's Function in a covariant form. The most obvious way to write something in a covariant form is to compose it of the inner-product of two 4-vectors. So, let's do some investigating with that in mind.

$$\delta(\Delta \mathrm{x}_\alpha \Delta \mathrm{x}^\alpha) = \delta\left[(x_0 - x_0{}')^2 - \vec{x}\bullet\vec{x}\right] \quad (85)$$

Here we have expanded an inner product in a delta-function to see where it will lead. We see, that under these circumstances we will acquire a g of the form:

$$g(x_0) = (x_0 - x_0{}')^2 - \vec{x} \bullet \vec{x} \tag{86}$$
$$= [(x_0 - x_0{}') - r][(x_0 - x_0{}') + r] \tag{87}$$
$$\tag{88}$$

This g has zeros at,

$$x_0 = x_0{}' + r \tag{89}$$
$$x_0 = x_0{}' - r \tag{90}$$
$$\tag{91}$$

Keep that in mind, as we meander in another direction and find that,

$$g'(x_0) = 2(x_0 - x_0{}') \tag{92}$$

Putting it all together, along with the identity, we find:

$$\delta(\Delta \mathrm{x}_\alpha \Delta \mathrm{x}^\alpha) = \frac{\delta[(x_0 - x_0{}') - r]}{|2(x_0{}' + r - x_0)|} + \frac{\delta[(x_0 - x_0{}') + r]}{|2(x_0{}' - r - x_0{}')|} \tag{93}$$
$$\Delta \mathrm{x}_\alpha = x_\alpha - x_\alpha{}' \tag{94}$$
$$\delta(\Delta \mathrm{x}_\alpha \Delta \mathrm{x}^\alpha) = \frac{1}{2r}\{\delta[(x_0 - x_0{}') - r] + \delta[(x_0 - x_0{}') + r]\} \tag{95}$$

Our covariant expression for our delta function in our Green's Function. This has one more term than we want, but that's okay because our Green's function has a step function in it, which will select the appropriate term for either the advanced or retarded Green's Function. Now, using this new-found-super-powerful function, we can express the 4-Potential. Which brings us to the starting point of chapter 14.1

$$A^\alpha(x) = A_{\mathrm{in}}{}^\alpha + \frac{4\pi}{c}\int G(x,x')J^\alpha(x')d^4x' \tag{96}$$

A_{in} is the homogeneous solution, which must be added to the inhomegeneous solution. And we have shown where the mysterious equation at the beginning 14.1 has come from. It was fairly ugly, but now we know the ins and outs of what we are working with. Now we would like to begin part 2 of this derivation. Find the potential due to a point charge.

VI. POINT CHARGE POTENTIAL

So here we are, once again at the beginning of Chapter 14. However, this time we have some questions answered. We know where A^α comes from. We also know the a little something about where G came from and how to use it. J^α is pretty much the same current density that we remember.

$$A^\alpha(x) = \frac{4\pi}{c}\int G_r(x,x')J^\alpha(x')d^4x' \tag{97}$$
$$J^\alpha(x') = \mathrm{ec}\int U^\alpha(\tau)\delta^4(x_\beta{}' - r(\tau)_\beta)d\tau \tag{98}$$
$$G_r(x,x') = \frac{H(x_0 - x_0{}')}{2\pi}\delta(\Delta \mathrm{x}_\alpha \Delta \mathrm{x}^\alpha) \tag{99}$$

If you take everything and put them together, you get an equation which is too long for the column.

$$A^\alpha(x) = \frac{4\pi}{c}\int \frac{H(x_0 - x_0{}')}{2\pi}\delta(\Delta \mathrm{x}_\mu \Delta \mathrm{x}^\mu)\mathrm{ec}... \tag{100}$$
$$...\int U^\alpha(\tau)\delta^4(x_\beta{}' - r(\tau)_\beta)d\tau d^4x'$$
$$A^\alpha(x) = 2e\int\int H(x_0 - x_0{}')\delta(\Delta \mathrm{x}_\mu \Delta \mathrm{x}^\mu)... \tag{101}$$
$$...U^\alpha(\tau)\delta^4(x_\beta{}' - r(\tau)_\beta)d^4x'd\tau$$

Now, we want to take the above mess and begin evaluating it for a point charge. Fortunately, right off the bat, we get to wipe out the quadruple integral over x', because we have a nice 4-dimensional delta function sitting right there,

$$A^\alpha(x) = 2e\int H(x_0 - r(\tau)_0)\delta\big((x_\mu - r(\tau)_\mu)^2\big)U^\alpha(\tau)d\tau \tag{102}$$

At this juncture, we are faced with a small problem. That highly-fancy and cunning covariant delta function looks like it's going to be hard to work with. But, let's take a minute to look at what purpose that delta function is serving in the scheme of things. The integral only contributes terms to the potential if,

$$\delta\big((x_\mu - r(\tau)_\mu)^2\big) = 1 \tag{103}$$

And that is only true if,

$$(x - r(\tau))_\mu(x - r(\tau))^\mu = 0 \tag{104}$$

Which expands to the equation for a cone:

$$(x_0 - r(\tau)_0)^2 - \big(\vec{x} - \vec{r}(\tau)\big)^2 = 0 \tag{105}$$
$$(x_0 - r(\tau)_0)^2 = \big(\vec{x} - \vec{r}(\tau)\big)^2 \tag{106}$$

This cone is 'The Light Cone.' This is saying that only charges residing on an objects light cone can contribute to the potential that the object feels. This is saying that the information about changes in a particles state, are transmitted at the speed of light. But so far, there is nothing in this cone to indicate that the transmissions are either forward or backward in time. We hope that causality is conserved, and thanks to that clunky step function, it is:

$$H(x_0 - r(\tau)_0) \rightarrow x_0 > (r(\tau_0))_0 \tag{107}$$

So, okay, it's great that we can make some sense about the physical meaning behind this piece of the equation, but it would be even nicer if we could figure out a way to progress through the derivation from this juncture. Well, we can progress, following in the footsteps of Jackson. However, I believe there is a flaw in his method. He lights into the equation again, utilizing the delta function identity from earlier.

$$\delta(g(x)) = \sum \frac{\delta(x_{\text{io}})}{|g'(x_{\text{io}})|} \tag{108}$$

And, just like before he sets g to be,

$$g(\tau) = (x - r(\tau))_\mu (x - r(\tau))^\mu \tag{109}$$

$$g(\tau) = (x - r(\tau))^2 \tag{110}$$

But, this time he has a repeat root! Because he chose to make his function τ instead - which is akin to making the $r(\tau)$ be his variable. And yet, he proceeds to find g'.

$$\frac{dg(\tau)}{\mathrm{d}\tau} = -2\frac{dr(\tau)}{\mathrm{d}\tau} \bullet (x - r(\tau)) = -2\mathrm{V}\bullet(x - r(\tau)) \tag{111}$$

Then, the roots of g, occur for some τ, for which $(x - r(\tau) = 0)$.

$$r^\alpha(\tau_0) = x^\alpha \tag{112}$$

Which leads to what seems to be a glaring mistake to me, perhaps I'm missing a subtlety here,

$$\delta(g(\tau)) = \frac{\delta(\tau - \tau_0)}{-2\mathrm{U}\bullet(x - r(\tau_0))} \tag{113}$$

In any case, once you have accepted the previous step, it easily leads to,

$$A^\alpha(x) = 2e \int \frac{\delta(\tau - \tau_0)}{-2\mathrm{U}\bullet(x - r(\tau_0))} U^\alpha(\tau) d\tau \tag{114}$$

$$A^\alpha(x) = \frac{-\mathrm{eU}^\alpha(\tau)}{\mathrm{U}\bullet(x - r(\tau_0))} \tag{115}$$

Since we are constrained to the light cone,

$$(x - r(\tau_0))^2 = 0 \tag{116}$$

$$(x_0 - r(\tau_0)_0)^2 = \left(\vec{x} - \vec{r}(\tau_0)\right)^2 \tag{117}$$

$$x_0 - r(\tau_0)_0 = \left|\vec{x} - \vec{r}(\tau_0)\right| = R \tag{118}$$

We can easily proceed to the classical forms of the Lienard Wiechert Potentials:

$$A^\alpha(x) = \frac{-\mathrm{eU}^\alpha(\tau)}{U_0(x_0 - r(\tau_0)_0) - \vec{U} \bullet \left(\vec{x} - \vec{r}(\tau_0)\right)} \tag{119}$$

$$A^\alpha(x) = \frac{-\mathrm{eU}^\alpha(\tau)}{\gamma \mathrm{cR}\left(1 - \vec{\beta} \bullet \hat{n}\right)} \tag{120}$$

$$A^\alpha(x) = \left(\phi, \vec{A}\right) = \frac{-e}{\gamma \mathrm{cR}\left(1 - \vec{\beta} \bullet \hat{n}\right)} \left(U_0, \vec{U}\right) \tag{121}$$

$$\phi = \frac{-\mathrm{eU}_0}{\gamma \mathrm{cR}\left(1 - \vec{\beta} \bullet \hat{n}\right)} = \frac{-e}{R\left(1 - \vec{\beta} \bullet \hat{n}\right)} \tag{122}$$

$$\vec{A} = \frac{-e\vec{U}}{\gamma \mathrm{cR}\left(1 - \vec{\beta} \bullet \hat{n}\right)} = \frac{-e\vec{\beta}}{R\left(1 - \vec{\beta} \bullet \hat{n}\right)} \tag{123}$$

And these are exactly the same results we derived in about five very easy steps at the beginning of the paper. Notice behavior as $c >> v$, the formulas for ϕ and A reduce to,

$$\phi = -\frac{e}{R} \tag{124}$$

$$\vec{A} = 0 \tag{125}$$

VII. FIELDS

Jackson then continues to derive the field expressions. I think, with the techniques and details gone through thusfar in the paper, and also considering that the paper is already 16 pages of mostly math, that we can safely skip the details of the derivation - owing to the fact that it will mostly be rehashing technqiues and identities already covered. Instead, I would like to present the field equations,

$$\vec{B} = [\hat{n} \times \vec{E}] \tag{126}$$

$$\vec{E} = e\left[\frac{\hat{n} - \vec{\beta}}{\gamma^2(1 - \vec{\beta} \cdot \hat{n})^3 R^2}\right] + \frac{e}{c}\left[\frac{\hat{n} \times [(\hat{n} - \vec{\beta}) \times \dot{\vec{\beta}}]}{(1 - \vec{\beta} \cdot \hat{n})^3 R}\right] \tag{127}$$

We can see very clearly from the $\vec{B}$ that the magnetic field is always perpendicular to the electric field and to the direction to the 'image' of the source charge. Furthermore, in $\vec{E}$, we can see that it's broken into two different powers of R. The first term dominates at short distance, the second term dominates far away. The first term is called the velocity field, because it only has a depenence on velocity and when the velocity and acceleration of the electron are very small, this term coalesces into the electric field due to a point charge. As is shown later in the chapter, the term that dominates in the long range and is also dependent on the acceleration is responsible for the radiation fields. Because of this, it is often called the radiation field, or due to its dependence on $\dot{\beta}$ the acceleration field.

VIII. CONCLUSION

I have thoroughly gone through the derivation of the Electrodynamic Green's function as well as the potentials due to a point charge moving in an arbitrary manner (Lienard Wiechert Potentials). I have done it in an intuitive manner, which can be immediately applied to actual point charges moving around and I have also followed in the more general footsteps of Jackson, and created a procedure which can be used as a guide for finding the potentials for more complicated moving systems using the Green's Function. Unfortunately, the small chapter that I covered, held mostly mathemtatical puzzles and very few physical or conceptual problems.

[1] Ch 12, Ch 14 John Jackson, Classical Electrodynamics, 3rd Edition

[2] Ch 10, David Griffiths, Introduction to Electrodynamics, 3rd Edition

[3] Ch 2, Ch 6, Ch 7, Appdx C, Susan M. Lea, Mathematics for Physicists